

A Recursion for the Farey Sequence Sequence

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1 A Recursion for the Farey Sequence

Represent a rational number f as an ordered pair (n, d) where n is the numerator and d is the denominator. The Farey sequence of order m , F_m , is given by the recursion

$$f_{i+1} = \left[\frac{d_{i-1} + m}{d_i} \right] f_i - f_{i-1}$$

where $f_1 = (0, 1)$ and $f_2 = (1, m)$ ([Har02], [Far16], [Hal70]).

The Farey sequence F_m is the sequence of irreducible rational numbers $\{\frac{n}{d}\}$ with $0 \leq n \leq d \leq m$ arranged in increasing order. It is occasionally useful to have available an expression for F_{m+1} in terms of F_m .

2 A Recursion for the Farey Sequence Sequence

Represent an element f of the Farey sequence F_m as an ordered triple (n_f, d_f, s) where n_f is the numerator, d_f is the denominator and s is the number of Farey sequences following F_m until a new fraction is created immediately after f . When the context is clear, the subscript f will be omitted.

In this notation F_2 is given by

$$F_2 = (0, 1, 1) || (1, 2, 1) || (1, 1, 0).$$

Starting with the null sequence, F_{m+1} may be computed from F_m as follows:

For i from 1 to $|F_m| - 1$

If $(s_i > 1)$

$$F_{m+1} \Leftarrow F_{m+1} || (n_i, d_i, s_i - 1)$$

else

$$F_{m+1} \Leftarrow F_{m+1} || (n_i, d_i, d_i) || (n_i + n_{i+1}, d_i + d_{i+1}, d_{i+1})$$

Next i

$$F_{m+1} \Leftarrow F_{m+1} || (1, 1, 0)$$

We shall refer to the second (mediant) fraction in the *else* clause as a *created* fraction and denote the set of all created fractions in the transition from F_m to F_{m+1} by C_m . The number of elements in C_m is $\varphi(m+1)$ where φ is the Euler totient function. We shall denote the (initial) value of s of a created fraction f by s_f .

Note that if $(n_d, d, 1) \in F_m$ for $1 \leq d \leq m$ then $|C_m| = \varphi(m+1) = m$ and $m+1$ is a prime.

3 Properties

The following are some elementary properties of the Farey sequences generated using the above notation:

- (1) F_m is the Farey sequence of order m
 $\Rightarrow (n, d, s) \in C_m \Leftrightarrow d = m+1, n < d \text{ and } n \nmid d.$
- (2) $(n_i, d, s_i), (n_j, d, s_j) \in F_m$ and $i \neq j \Rightarrow s_i \neq s_j.$

- (3) $(n, d, s_f) \in C_{m'}$ and $(n, d, s) \in F_m$ for $m \geq m' + s_f$
 $\Rightarrow (n, d, s) \in F_{m+kd}$ for $k = 1, 2, \dots$
- (4) $f = (n, d, s) \in F_m \Rightarrow s = d - (m - s_f) \bmod d$
- (5) $f = (n, d, 1) \in C_m \Leftrightarrow (m - s_f + 1) \bmod d = 0$
- (6) $(n_i, d_i, 1), (n_j, d_j, 1) \in F_m$ with $i \neq j \Rightarrow d_i \neq d_j$
- (7) $m + 1$ is prime $\Leftrightarrow (n_i, i, 1) \in F_m$ for $i = 1, \dots, m$

These properties may be interpreted as follows:

(1) says that if F_m is the Farey sequence of order m then F_{m+1} is the Farey sequence of order $m + 1$.

(2) says that all fractions with the same denominator in a given Farey sequence have different s values and in particular the s_f values of elements of C_m all different.

(3) says that after an initial period of length s_f the s value associated with a fraction with denominator d cycles with a period of length d .

(4) is a restatement of (3).

(5) is a special case of (4).

(6) says that the denominators of the fractions immediately preceding created fractions are all different. As the sum of the preceding and succeeding denominators of a created fraction is $m + 1$, the same holds for the succeeding denominators.

(7) is the special case of (6) where $m + 1$ is a prime.

4 Functions of the Farey Sequence Sequence

The above recursion for generating the sequence of Farey sequences may be used to describe properties of Farey sequences as a function of the order of the sequence.

For example, the distance between a Farey fraction $f = (n, d, s)$ in F_m and the next larger Farey fraction in F_m as a function of m is given by

$$\left(\left[\frac{m - s_f}{d} \right] d^2 + s_f d \right)^{-1}.$$

As another example, one can express the order index of an element f of the Farey sequence F_m as a function of F_m and m as follows. Let i_f denote the (initial) order index of f in F_{d_f} . Then the order index of f in F_m for $m > d_f$ is given by

$$I_m(f) = i_f + \sum_{g < f} \left[\frac{m - s_g}{d_g} \right] - \left[\frac{d_f - s_g}{d_g} \right]$$

The result of Franel and Landau [FL24] regarding the Riemann Hypothesis and the Farey series may be written using the above notation as

$$\sum_{f \in F_m} \left(\frac{I_m(f)}{\Phi(m) + 1} - f \right)^2 = \mathcal{O}(m^{-1+\epsilon})$$

where Φ is the Euler summatory function.

5 Cycles

The above recursion for generating the sequence of Farey sequences may also be used to generate sequences of primes. For this purpose, we develop notation to describe the cycling behavior of s .

If

$$m_c(f, k) = \begin{cases} kd_f + s_f - c & c \leq s_f \\ (k+1)d_f + s_f - c & c > s_f \end{cases}$$

then

$$M_c(f) = \{m_c(f, k), k = 1, 2, \dots\}$$

is the set of all m such that the value of s associated f in F_m is c . Further, if

$$\mathcal{M}_c(d) = \bigcup_{d_f=d} M_c(f),$$

then $\mathcal{M}_c(d)$ is the set of all m such that the value of s associated with a Farey fraction in F_m with denominator d is c . The $\mathcal{M}_c(d)$ sets can be to define prime sieves and prime recursions.

6 Prime Sieves

p is a prime if

$$p-1 \in \bigcap_{d=1}^{p-1} \mathcal{M}_1(d)$$

This computation is equivalent to ensuring that there exists an $f \in C_d$ such that

$$\left\langle \frac{p-s_f}{d_f} \right\rangle = 0$$

for each $1 \leq d \leq p-1$.

p is the lesser of a twin prime [Guy04] if

$$p-1 \in \bigcap_{d=1}^{p-1} \left\{ \mathcal{M}_1(d) \bigcap \mathcal{M}_3(d) \right\}$$

p is a prime since

$$p-1 \in \bigcap_{d=1}^{p-1} \mathcal{M}_1(d).$$

Further since $([p/2], p, 2) \in F_p$ we have $([p/2], p, 1) \in F_{p+1}$ along with $(0, 1, 1)$ and $(p, p+1, 1)$. The condition

$$p-1 \in \bigcap_{d=1}^{p-1} \mathcal{M}_3(d)$$

ensures that $(n_d, d, 1) \in F_{p+1}$ for $2 \leq d \leq p-1$. Therefore $(n_d, d, 1) \in F_{p+1}$ for $1 \leq d \leq p+1$ and hence $p+2$ is a prime.

7 Prime Recursions

Let $m_3 = \mathcal{M}_1(2)$ and $d_3 = 3$. Set

$$m_{i+1} = m_i \bigcap \mathcal{M}_1(d_i)$$

and

$$d_{i+1} = 1 + \min m_{i+1}.$$

Then d_i is the i^{th} prime.

Let $(p, p+2)$ be a twin prime and let

$$q = 1 + \min \bigcap_{d=1}^p \left\{ \mathcal{M}_1(d) \bigcap \mathcal{M}_3(d) \right\}.$$

Then $(q, q+2)$ is the next twin prime. This recursion can be expressed incrementally as follows.

Set $m_1 = \mathcal{M}_1(3) \bigcap \mathcal{M}_3(3)$ and $d_1 = 3$. Let

$$d_{i+1} = 1 + \min m_i$$

and

$$m_{i+1} = m_i \bigcap \left\{ \bigcap_{d=d_i+1}^{d_{i+1}} \mathcal{M}_1(d) \bigcap \mathcal{M}_3(d) \right\}$$

Then $\{d_i, d_i+2\}$ is the i^{th} prime pair.

8 Twin Prime Generation

The following is a Mathematica module to generate the sequence of twin primes.

```
TwinPrimes[twins_, k_] := Module[{i, j, l, p, q},
  l = Intersection[Ems[3,1,k], Ems[3,3,k]];
  p = 3;
  q = Min[l] + 1;
  For[i = 1, i < twins - 1, i++,
    Print["Twin Pair #", i, ": ", {p, p + 2}];
    For[j = p + 1, j <= q, j++,
      l = Intersection[l, Ems[j,1,k], Ems[j,3,k]];
    ];
    p = q;
    q = Min[l] + 1;
  ]
  Print["Twin Pair #", i, ": ", {p, p + 2}];
  Print["Twin Pair #", i + 1, ": ", {q, q + 2}];
]
```

where $\text{Ems}[d, c, k]$ is

$$\bigcup_{d_f=d} \{m_c(f, k), k = 1, 2, \dots, k\}$$

9 Software

A Mathematica notebook implementing all the above computations is available from the author upon request.

References

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